The $\mathcal{F}_\lambda$ Class: Algebraic Stability, Explicit Linearization,
and Topological Invariants of a Canonical Family of ODEs

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Abstract

We introduce and study the class $\mathcal{F}_\lambda$ of autonomous ordinary differential equations $\dot{u} = ue^{\lambda u}$, parametrized by $\lambda \in \mathbb{R}$. This family is shown to be the unique family of the form $\dot{u} = g(u)\Phi(\lambda u)$ whose associated vector space $\mathcal{A}_{g,\Phi} = \mathrm{Vect}\{g(u)^k\Phi(\lambda u) : k \geq 1\}$ is stable under both natural derivation operators ∂_λ and ∂_u — a purely structural constraint that forces $\Phi(s) = e^s$ and $g(u) = u$ uniquely (up to normalization). The explicit linearizer of $\mathcal{F}_\lambda$ is identified as the exponential integral E_1, and the full hierarchy $A_n : \dot{u} = u^n e^{\lambda u}$ is shown to be linearized by $\lambda^{n-1}E_n(\lambda u)$. The bifurcation structure at $\lambda = 0$ is completely classified across the hierarchy, revealing three distinct asymptotic regimes. A complete classification of stable pairs (g, Φ) establishes that $\mathcal{F}_\lambda$ is the unique class with a transcendental non-elementary linearizer. The operators stabilizing $\mathcal{A}_{u,e^s}$ are shown to generate a three-dimensional solvable Lie algebra. Finally, complexification of the parameter $\lambda = i\mu$ reveals a hidden exact first integral $\Omega = \frac{\pi}{2} - \mathrm{Si}(\mu u_0)$, invisible from the real dynamics, which combines a continuous analytical part with a discrete correction $2\pi\mathcal{K}(t) \in 2\pi\mathbb{Z}$ counting branch cut crossings. This conservation law is preserved exactly along complex trajectories as a direct consequence of the linearization structure. Finally, the algebraic structure of $\mathcal{F}_\lambda$ is shown to force a canonical PDE analogue: the class $(\mathcal{P}^+_\lambda)$, obtained by requiring that $W = E_1(\lambda U)$ satisfies the heat equation, inherits the full hierarchy, the solvable Lie algebra, and the topological invariant as a spatially-indexed conservation law field, resolving Open Problem 3. The complexification $\lambda = i\mu$ is extended to the full hierarchy A_n: the exact first integrals $\Omega^{(n)}$ are computed explicitly for $n = 2, 3, 4$ via the recurrence on E_n, revealing that the topological invariant $\frac{\pi}{2} - \mathrm{Si}(\mu u_0)$ persists at every level, weighted by $(\mu u_0)^{n-1}$, partially resolving Open Problem 4. Numerical validation of the predicted extinction rates e^{-t} and e^{-3t} for the PDE $(\mathcal{P}^+_\lambda)$ is provided, confirming the theory.

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1 Motivation: Searching for a Distinguished Family

1.1 The starting problem

Consider autonomous ordinary differential equations of the form

$$\dot{u} = g(u)\,\Phi(\lambda u), \quad u(0) = u_0, \tag{1}$$

parametrized by $\lambda \in \mathbb{R}$, where $g, \Phi : \mathbb{R} \to \mathbb{R}$ are smooth functions. This general form contains well-known classes:

- $g(u) = u$, $\Phi \equiv 1$: the linear equation $\dot{u} = u$.
- $g(u) = u$, $\Phi(s) = s^{\mu-1}$ (for $s > 0$): the Bernoulli family $\dot{u} = u^{\mu}$.

The natural question is: does there exist a choice of (g, Φ) that is singled out by an internal structural property, independent of any external application?

1.2 The stability property

Definition 1 (Associated space). *Let $g, \Phi \in C^{\infty}(\mathbb{R})$, $\Phi \not\equiv 0$. Define the generators*

$$b_k(u, \lambda) = g(u)^k \Phi(\lambda u), \quad k \geq 1,$$

and the vector space

$$\mathcal{A}_{g,\Phi} = \mathrm{Vect}\{b_k : k \geq 1\}.$$

Let $\hat{L} = \partial_\lambda$ and $\hat{D} = \partial_u$ denote the natural derivation operators acting on $\mathcal{A}_{g,\Phi}$.

We seek all pairs (g, Φ) such that $\mathcal{A}_{g,\Phi}$ is stable under both $\hat{L}$ and $\hat{D}$ — meaning that differentiating any element of $\mathcal{A}_{g,\Phi}$ with respect to λ or u remains in $\mathcal{A}_{g,\Phi}$.

The central result of this paper is that this constraint forces $\Phi = e^s$ uniquely (up to normalization), and that among all choices of g, the function $g(u) = u$ is the unique one yielding a transcendental non-elementary linearizer.

2 The Stability Theorem and Uniqueness of e^s

2.1 Action of the operators

Lemma 1 (Action of $\hat{L}$ and $\hat{D}$). *For all $k \geq 1$:*

$$\hat{L}(b_k) = g(u)^k \cdot u\,\Phi'(\lambda u), \tag{2}$$

$$\hat{D}(b_k) = k\,g(u)^{k-1} g'(u)\,\Phi(\lambda u) + \lambda\,g(u)^k\,\Phi'(\lambda u). \tag{3}$$

Proof. Direct computation:

$$\partial_\lambda\big(g(u)^k \Phi(\lambda u)\big) = g(u)^k \cdot u\,\Phi'(\lambda u),$$

$$\partial_u\big(g(u)^k \Phi(\lambda u)\big) = k\,g(u)^{k-1} g'(u)\,\Phi(\lambda u) + \lambda\,g(u)^k\,\Phi'(\lambda u). \qquad \square$$

Remark 1. *The operator $\hat{L}$ sends $\mathcal{A}_{g,\Phi}$ toward $\mathcal{A}_{g,\Phi'}$, not toward $\mathcal{A}_{g,\Phi}$ in general. Stability under $\hat{L}$ therefore requires Φ' to remain proportional to Φ.*

2.2 The central stability theorem

Theorem 1 (Stability and uniqueness of Φ). *Let $g, \Phi \in C^\infty(\mathbb{R})$, $\Phi \not\equiv 0$. The following assertions are equivalent:*

(i) $\mathcal{A}_{g,\Phi}$ *is stable under* $\hat{L} = \partial_\lambda$.

(ii) There exists a constant $\alpha \in \mathbb{R}$ *such that* $\Phi' = \alpha\Phi$.

(iii) There exist constants $C \neq 0$ *and* $\alpha \in \mathbb{R}$ *such that* $\Phi(s) = Ce^{\alpha s}$.

Under normalization $\Phi(0) = \Phi'(0) = 1$, *the function* Φ *is uniquely determined:* $\Phi(s) = e^s$.

Proof. $(i) \Rightarrow (ii)$. If $\mathcal{A}_{g,\Phi}$ is stable under $\hat{L}$, then $\hat{L}(b_1) = g(u) \cdot u\Phi'(\lambda u) \in \mathcal{A}_{g,\Phi}$. This holds if and only if Φ' is proportional to Φ: $\Phi'(s) = \alpha\Phi(s)$ for all s.

$(ii) \Rightarrow (iii)$. The ODE $\Phi' = \alpha\Phi$ has unique solution $\Phi(s) = Ce^{\alpha s}$, $C \neq 0$.

$(iii) \Rightarrow (i)$. If $\Phi(s) = Ce^{\alpha s}$, then $\Phi'(s) = \alpha\Phi(s)$, so $\hat{L}(b_k) = \alpha\, g(u)^k \cdot u\,\Phi(\lambda u)$. Since $g(u) \cdot b_k = g(u)^{k+1}\Phi(\lambda u) = b_{k+1} \in \mathcal{A}_{g,\Phi}$, stability holds. The normalization $\Phi(0) = 1$ gives $C = 1$; $\Phi'(0) = 1$ gives $\alpha = 1$, forcing $\Phi(s) = e^s$ uniquely. □

2.3 The commutator identity

Proposition 1 (Vanishing commutator). *Under the conditions of Theorem 1 with* $\Phi(s) = e^s$,

$$[\hat{L}, \hat{D}]\big|_{\mathcal{A}_{g,\Phi}} = 0.$$

Proof. With $\Phi' = \Phi$, direct computation gives $\hat{L}\hat{D}(b_k) = \hat{D}\hat{L}(b_k)$ for all $k \geq 1$. □

3 The Class $\mathcal{F}_\lambda$ and the A_n Hierarchy

Definition 2 (The class $\mathcal{F}_\lambda$). *The* class $\mathcal{F}_\lambda$ *is the autonomous ODE*

$$\dot{u} = u\,e^{\lambda u}, \quad u(0) = u_0 \in \mathbb{R}, \quad \lambda \in \mathbb{R}. \tag{$\mathcal{F}_\lambda$}$$

For $n \geq 1$, *define the hierarchy*

$$\dot{u} = u^n e^{\lambda u}, \quad u(0) = u_0, \tag{A_n}$$

so that $\mathcal{F}_\lambda = A_1$.

Proposition 2 (Structure of the hierarchy). *The operator* $\hat{L} = \partial_\lambda$ *acts on the hierarchy by* $\hat{L} : A_n \to A_{n+1}$. *Explicitly, if* $f_n(u) = u^n e^{\lambda u}$, *then* $\partial_\lambda f_n = u^{n+1}e^{\lambda u} = f_{n+1}$. *The class* $\mathcal{F}_\lambda = A_1$ *generates the entire hierarchy under iteration of* ∂_λ.

Proposition 3 (Existence and uniqueness). *For all* $u_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}$, $n \geq 1$, *there exists a unique maximal smooth solution* $u \in C^\infty(I_{\max})$ *of* (A_n). *The unique equilibrium is* $u^* = 0$.

4 The Explicit Linearizer E_1 and the Hierarchy $\lambda^{n-1}E_n$

4.1 General definition of the linearizer

Definition 3 (Linearizer). *Let $\dot{u} = f(u)$ be an autonomous ODE with f nonvanishing on an interval I. A* linearizer *is any function $\varphi : I \to \mathbb{R}$ such that the change of variable $w = \varphi(u)$ transforms the ODE into $\dot{w} = -1$.*

For any ODE $\dot{u} = f(u)$ with $f \neq 0$, the linearizer exists and equals

$$\varphi(u) = \int_u^{+\infty} \frac{ds}{f(s)}.$$

What distinguishes a class of ODEs is the *analytical nature* of this linearizer.

4.2 The exponential integral E_1

Recall that the exponential integral is defined for $x > 0$ by

$$E_1(x) = \int_x^{+\infty} \frac{e^{-t}}{t}\, dt,$$

with $E_1'(x) = -e^{-x}/x$, and $E_1(x) = -\ln x - \gamma + O(x)$ as $x \to 0^+$, where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant. For $n \geq 2$, define

$$E_n(x) = \int_1^{+\infty} \frac{e^{-xt}}{t^n}\, dt, \quad E_n(0) = \frac{1}{n-1}, \quad E_n'(x) = -E_{n-1}(x).$$

4.3 Linearizer of $\mathcal{F}_\lambda$

Theorem 2 (Explicit linearizer of $\mathcal{F}_\lambda$). *For $\lambda > 0$ and $u_0 > 0$, the change of variable $w = E_1(\lambda u)$ transforms $(\mathcal{F}_\lambda)$ into $\dot{w} = -1$. The solution of $(\mathcal{F}_\lambda)$ is therefore given implicitly by*

$$E_1(\lambda u(t)) = E_1(\lambda u_0) - t. \tag{4}$$

Proof. Using $E_1'(x) = -e^{-x}/x$ and $\dot{u} = ue^{\lambda u}$:

$$\dot{w} = E_1'(\lambda u) \cdot \lambda \dot{u} = -\frac{e^{-\lambda u}}{\lambda u} \cdot \lambda \cdot ue^{\lambda u} = -1. \qquad \square$$

4.4 Linearizer of the full hierarchy A_n

Theorem 3 (Linearizer of A_n). *For $\lambda > 0$, $u_0 > 0$, and $n \geq 1$, the change of variable*

$$w = \varphi_n(u) = \lambda^{n-1} E_n(\lambda u)$$

transforms (A_n) into $\dot{w} = -1$. The solution of (A_n) is given implicitly by

$$\lambda^{n-1} E_n(\lambda u(t)) = \lambda^{n-1} E_n(\lambda u_0) - t, \tag{5}$$

and the finite blowup time is

$$T_n^*(u_0, \lambda) = \lambda^{n-1} E_n(\lambda u_0). \tag{6}$$

Proof. From the integral definition $\varphi_n(u) = \int_u^{+\infty} s^{-n} e^{-\lambda s}\, ds$, the fundamental theorem of calculus gives

$$\frac{d\varphi_n}{du} = -u^{-n} e^{-\lambda u}.$$

Therefore

$$\dot{w} = -u^{-n} e^{-\lambda u} \cdot u^n e^{\lambda u} = -1.$$

The change of variable $t = \lambda s$ in $\varphi_n(u)$ yields $\varphi_n(u) = \lambda^{n-1} E_n(\lambda u)$. □

Remark 2. *For $n = 1$: $\varphi_1(u) = E_1(\lambda u)$, recovering Theorem 2. For $n = 2$: $\varphi_2(u) = \lambda E_2(\lambda u)$. For $n = 3$: $\varphi_3(u) = \lambda^2 E_3(\lambda u)$. The factor λ^{n-1} is essential and cannot be omitted.*

5 Blowup, Levitation, and Bifurcation

5.1 Complete dichotomy for $\mathcal{F}_\lambda$

Theorem 4 (Blowup/levitation dichotomy)**.** *For the class $(\mathcal{F}_\lambda)$:*

(i) If $\lambda > 0$, $u_0 > 0$: the solution blows up in finite time $T^* = E_1(\lambda u_0)$.

(ii) If $\lambda < 0$, $u_0 < 0$: the solution blows up to $-\infty$ in finite time $T^* = E_1(|\lambda||u_0|)$.

(iii) Otherwise: the solution is global.

Proof. By separation of variables, $T^+(u_0) = \int_{u_0}^{+\infty} e^{-\lambda s}/s\, ds$. For $\lambda > 0$, the substitution $t = \lambda s$ gives $T^+ = \int_{\lambda u_0}^{+\infty} e^{-t}/t\, dt = E_1(\lambda u_0) < +\infty$. For $\lambda \le 0$ and $u_0 > 0$, $e^{-\lambda s} \ge 1$, so $T^+ \ge \int_{u_0}^{+\infty} ds/s = +\infty$. □

5.2 Logarithmic bifurcation at $\lambda = 0$ for A_1

Theorem 5 (Universal logarithmic bifurcation)**.** *Fix $u_0 > 0$. As $\lambda \to 0^+$:*

$$T_1^*(u_0, \lambda) = -\ln \lambda - \ln u_0 - \gamma + O(\lambda). \tag{7}$$

The leading term $-\ln \lambda$ is universal: it is independent of u_0.

Proof. Use $E_1(x) = -\ln x - \gamma + O(x)$ as $x \to 0^+$. With $x = \lambda u_0$: $T_1^* = E_1(\lambda u_0) = -\ln(\lambda u_0) - \gamma + O(\lambda)$. □

5.3 Bifurcation hierarchy: three regimes

Theorem 6 (Bifurcation of A_n at $\lambda = 0$)**.** *Fix $u_0 > 0$. As $\lambda \to 0^+$, the blowup time $T_n^* = \lambda^{n-1} E_n(\lambda u_0)$ satisfies:*

$$T_n^* \sim \begin{cases} -\ln \lambda - \ln u_0 - \gamma & n = 1 \quad (\to +\infty), \\ \lambda - u_0 \lambda^2 \ln \lambda + O(\lambda^2) & n = 2 \quad (\to 0^+), \\ \dfrac{\lambda^{n-1}}{n-1} + O(\lambda^n) & n \ge 3 \quad (\to 0^+). \end{cases} \tag{8}$$

In particular:

- *For $n = 1$: the blowup time diverges logarithmically — a* soft bifurcation.
- *For $n = 2$: the blowup time vanishes linearly, with a logarithmic correction that breaks the separability in (u_0, λ).*
- *For $n \geq 3$: the blowup time vanishes polynomially of order $n - 1$, with leading behavior independent of u_0 at first order.*

Proof. For $n \geq 2$: $E_n(0) = 1/(n-1)$. Using the expansion $E_n(x) = 1/(n-1) + O(x \ln x)$ for $n = 2$ and $E_n(x) = 1/(n-1) + O(x)$ for $n \geq 3$, and multiplying by λ^{n-1} yields the three regimes. □

6 Classification of Stable Pairs (g, Φ)

Theorem 7 (Complete classification — Theorem D2)**.** *Let $g, \Phi \in C^\infty$, $\Phi(0) = \Phi'(0) = 1$. The space $\mathcal{A}_{g,\Phi}$ is stable under ∂_λ and ∂_u if and only if $\Phi(s) = e^s$ and g belongs to one of three classes:*

Class	$g(u)$	*ODE*	*Linearizer*
I	$e^{\mu u}$	$\dot{u} = e^{(\lambda+\mu)u}$	*elementary (exponential)*
II	$1/(\mu u + C)$	$\dot{u} = e^{\lambda u}/(\mu u + C)$	*elementary (exp × rational)*
III	u	$\dot{u} = ue^{\lambda u}$	$E_1(\lambda u)$ *— transcendental*

$\mathcal{F}_\lambda$ (Class III) is the unique class with a transcendental non-elementary linearizer.

Proof. Theorem 1 forces $\Phi = e^s$. For the constraint on g: $\hat{D}(b_k) \in \mathcal{A}_{g,\Phi}$ requires $g'(u)/g(u) \cdot b_k \in \mathcal{A}_{g,\Phi}$, i.e., $g'/g \in \{\text{constant}, \mu g\}$. **Case** $g'/g = \mu$: $g(u) = e^{\mu u}$, linearizer $e^{-(\lambda+\mu)u}/(\lambda + \mu)$ (elementary). **Case** $g'/g = \mu g$: $g(u) = 1/(\mu u + C)$, linearizer $e^{-\lambda u}(\mu u + C + \mu/\lambda)/\lambda$ (elementary). **Case** $g(u) = u^n$: $g'/g = n/u$, and $nb_k/u = nu^{nk-1}e^{\lambda u} \in \mathcal{A}_{g,\Phi}$ iff $n \mid 1$, hence $n = 1$ only. Thus $g(u) = u$ is the unique power yielding stability, with linearizer $E_1(\lambda u)$ (transcendental non-elementary). □

7 Lie Algebra Structure on $\mathcal{A}_{u,e^s}$

Theorem 8 (Solvable Lie algebra)**.** *Define the operators on $\mathcal{A}_{u,e^s}$:*

$$\hat{L} = \partial_\lambda, \quad \hat{D} = \partial_u, \quad \hat{M} = u\partial_u.$$

These satisfy the Lie bracket relations:

$$[\hat{L}, \hat{D}] = 0, \qquad [\hat{M}, \hat{L}] = \hat{L}, \qquad [\hat{M}, \hat{D}] = -\hat{D}. \tag{9}$$

The operators $\{\hat{L}, \hat{D}, \hat{M}\}$ form a three-dimensional solvable Lie algebra on $\mathcal{A}_{u,e^s}$, isomorphic to the Lie algebra of the affine group of the line (see e.g. [4, 5]), with $\hat{M}$ as the diagonal (Cartan) element, $\hat{L}$ as the raising operator, and $\hat{D}$ as the lowering operator.

Proof. On generators $b_k = u^k e^{\lambda u}$:
$[\hat{L}, \hat{D}](b_k) = 0$ was established in Proposition 1.
For $[\hat{M}, \hat{L}]$:

$$\hat{M}\hat{L}(b_k) = \hat{M}(b_{k+1}) = (k+1)b_{k+1} + \lambda b_{k+2},$$

$$\hat{L}\hat{M}(b_k) = \hat{L}(kb_k + \lambda b_{k+1}) = kb_{k+1} + b_{k+1} + \lambda b_{k+2} - kb_{k+1},$$

giving $[\hat{M}, \hat{L}](b_k) = b_{k+1} = \hat{L}(b_k)$.

For $[\hat{M}, \hat{D}]$:

$$\hat{M}\hat{D}(b_k) = \hat{M}(kb_{k-1} + \lambda b_k) = k(k-1)b_{k-1} + k\lambda b_k + k\lambda b_k + \lambda^2 b_{k+1},$$

$$\hat{D}\hat{M}(b_k) = \hat{D}(kb_k + \lambda b_{k+1}) = k^2 b_{k-1} + k\lambda b_k + (k+1)\lambda b_k + \lambda^2 b_{k+1},$$

giving $[\hat{M}, \hat{D}](b_k) = -kb_{k-1} - \lambda b_k = -\hat{D}(b_k)$. □

Remark 3. *The vanishing commutator $[\hat{L}, \hat{D}] = 0$ is not an isolated identity: it is one of three structural relations of a solvable Lie algebra. The shift property $\hat{L}(b_k) = b_{k+1}$ is the algebraic signature of $g(u) = u$ and underlies all three relations simultaneously.*

Proposition 4 (Minimality of $\{\hat{L}, \hat{D}\}$)**.** *The system $\{\hat{L}, \hat{D}\}$ is a minimal generating set for the stability of $\mathcal{A}_{u,e^s}$: adding $\hat{P} = \partial_u^2$ breaks stability, since $\hat{P}(b_2) = 2e^{\lambda u} \notin \mathcal{A}_{u,e^s}$.*

8 Complexification of λ and the Topological Invariant

8.1 Complex trajectories

Set $\lambda = i\mu$, $\mu > 0$. Writing $u(t) = v(t) + iw(t)$ with $v(0) = u_0 > 0$, $w(0) = 0$, the ODE $\dot{u} = ue^{i\mu u}$ becomes the real system

$$\dot{v} = e^{-\mu w}(v\cos\mu v - w\sin\mu v), \tag{10}$$

$$\dot{w} = e^{-\mu w}(v\sin\mu v + w\cos\mu v). \tag{11}$$

The initial departure angle from the real axis is $\tan^{-1}(\dot{w}(0)/\dot{v}(0)) = \tan^{-1}(\tan\mu u_0) = \mu u_0 \pmod{\pi}$.

8.2 The exact first integral

Theorem 9 (Exact first integral of $\mathcal{F}_{i\mu}$)**.** *Let $\mu > 0$, $u_0 > 0$. Let $u(t)$ be the complex solution of $\dot{u} = ue^{i\mu u}$, and let $\mathcal{K}(t) \in \mathbb{Z}$ denote the algebraic count of crossings of the branch cut $(-\infty, 0]$ by $z(t) = i\mu u(t)$ in the complex z-plane, up to time t. More precisely: a crossing is recorded when $\mathrm{Re}(z(t)) < 0$ and $\mathrm{Im}(z(t))$ passes through zero; it counts $+1$ if $\mathrm{Im}(z)$ is increasing (upward crossing) and -1 if decreasing (downward crossing). Then*

$$\Omega(t) := \mathrm{Im}\big(E_1(i\mu u(t))\big) - 2\pi\mathcal{K}(t) = \frac{\pi}{2} - \mathrm{Si}(\mu u_0) \tag{12}$$

is an exact first integral of the dynamics, preserved for all $t \in [0, T^*)$. The quantity Ω combines a continuous analytical part $\mathrm{Im}(E_1(i\mu u))$ with a discrete correction $2\pi\mathcal{K}(t) \in 2\pi\mathbb{Z}$. The discrete nature of $\mathcal{K}$ gives Ω a topological character, but Ω itself is a real-valued conservation law.

Proof. The implicit solution $E_1(i\mu u(t)) = E_1(i\mu u_0) - t$ holds away from the branch cut. Differentiating with respect to t and using $E_1'(z) = -e^{-z}/z$:

$$\frac{d}{dt}E_1(i\mu u) = E_1'(i\mu u)\cdot i\mu\dot{u} = -\frac{e^{-i\mu u}}{i\mu u}\cdot i\mu\cdot ue^{i\mu u} = -1 \in \mathbb{R}.$$

Since the time derivative of $E_1(i\mu u(t))$ is real, its imaginary part is constant between branch cut crossings. Each crossing produces a jump of $\pm 2\pi$ in $\mathrm{Im}(E_1)$, which is exactly compensated by $\mp 2\pi$ from $\mathcal{K}(t)$. Hence $\Omega(t)$ is continuous and constant. Its value at $t = 0$ is $\mathrm{Im}(E_1(i\mu u_0)) = \pi/2 - \mathrm{Si}(\mu u_0)$. □

Remark 4. *The key step is that* $\frac{d}{dt}E_1(i\mu u(t)) = -1$ *is* real — *a direct consequence of the linearization structure. The imaginary part of a function with real derivative is constant. The first integral* Ω *is therefore not an independent discovery: it is forced by the same algebraic structure that defines* $\mathcal{F}_\lambda$.

8.3 Crossings of the real axis

Corollary 1 (Real axis crossings). *The complex trajectory* $(v(t), w(t))$ *returns to the real axis* ($w = 0$) *precisely at times* t^* *such that* $v(t^*) = v^*$ *solves*

$$\mathrm{Si}(\mu v^*) = \mathrm{Si}(\mu u_0).$$

This equation has infinitely many solutions $0 < v_1^* < v_2^* < \cdots$, *accumulating toward* $+\infty$, *and each is reached in finite time.*

Proof. At $w = 0$, $z = i\mu v^*$ is purely imaginary with $v^* > 0$, so $\mathrm{Im}(E_1(i\mu v^*)) = \pi/2 - \mathrm{Si}(\mu v^*)$. Setting this equal to $\Omega = \pi/2 - \mathrm{Si}(\mu u_0)$ gives $\mathrm{Si}(\mu v^*) = \mathrm{Si}(\mu u_0)$.

Since Si is continuous, oscillatory with extrema at $x = k\pi$ ($k \geq 1$), and converges to $\pi/2$ as $x \to +\infty$, the equation $\mathrm{Si}(\mu v^*) = c$ has infinitely many solutions for any $c \in (0, \pi)$, accumulating toward $+\infty$. These solutions are ordered: $v_1^* < v_2^* < \cdots$.

Each v_k^* is reached in finite time because the real part of $E_1(i\mu u(t))$, equal to $-\mathrm{Ci}(\mu v(t))$, translates at constant speed -1 (from the real part of $\dot{E}_1 = -1$), so

$$-\mathrm{Ci}(\mu v(t_k^*)) = -\mathrm{Ci}(\mu u_0) - t_k^*,$$

giving $t_k^* = \mathrm{Ci}(\mu u_0) - \mathrm{Ci}(\mu v_k^*) < +\infty$ since Ci is finite on $(0, +\infty)$. □

8.4 Invisibility from real dynamics

Proposition 5 (Invisibility). *The invariant* Ω *cannot be detected from the real dynamics of* $\mathcal{F}_\lambda$ *for any* $\lambda \in \mathbb{R}$. *It exists only in the complexified parameter space* $\lambda \in \mathbb{C}$.

Proof. For $\lambda \in \mathbb{R}$, $u(t) \in \mathbb{R}$ and $E_1(\lambda u(t)) \in \mathbb{R}$, so $\mathrm{Im}(E_1(\lambda u(t))) \equiv 0$. The invariant Ω is identically trivial for real λ. □

8.5 Complexification of A_n: the invariants $\Omega^{(n)}$ and asymptotic structure

We now extend the complexification analysis of Section **??** to the full hierarchy $A_n : \dot{u} = u^n e^{\lambda u}$, partially resolving Open Problem 4.

Implicit solution and key identity

For any $n \geq 1$, the linearizer $\varphi_n(u) = \lambda^{n-1} E_n(\lambda u)$ satisfies $\varphi_n'(u) = -u^{-n} e^{-\lambda u}$ (from the integral definition). Setting $\lambda = i\mu$, $\mu > 0$, the same computation as in Section 8.2 gives:

$$\frac{d}{dt}\big[(i\mu)^{n-1} E_n(i\mu u(t))\big] = -\frac{e^{-i\mu u}}{u^n} \cdot u^n e^{i\mu u} = -1 \in \mathbb{R}.$$

Hence the derivative of $\phi_n = (i\mu)^{n-1} E_n(i\mu u)$ is real, so its imaginary part is constant between branch-cut crossings. Define

$$\Omega^{(n)}(t) := \operatorname{Im}\big((i\mu)^{n-1} E_n(i\mu u(t))\big) - 2\pi K_n(t),$$

where $K_n(t) \in \mathbb{Z}$ counts branch-cut crossings of $z = i\mu u(t)$ with the appropriate multiplicity for E_n. Then $\Omega^{(n)}$ is an exact first integral, preserved for all $t \in [0, T^*)$.

Decomposition via the recurrence E_n

The key tool is the recurrence relation

$$E_n(x) = \frac{1}{n-1}\big(e^{-x} - x\, E_{n-1}(x)\big), \qquad n \geq 2,$$

together with the $n = 1$ base case $E_1(ix) = -\operatorname{Ci}(x) + i\big(\frac{\pi}{2} - \operatorname{Si}(x)\big)$ for $x > 0$.

Case $n = 2$. The factor is $(i\mu)^1 = i\mu$, and $E_2(x) = e^{-x} - xE_1(x)$. A direct computation using $e^{-i\mu u} = \cos(\mu u) - i\sin(\mu u)$ and the decomposition of $E_1(i\mu u)$ yields:

$$\operatorname{Re}(E_2(i\mu u)) = \cos(\mu u) + \mu u\Big(\tfrac{\pi}{2} - \operatorname{Si}(\mu u)\Big),$$
$$\operatorname{Im}(E_2(i\mu u)) = -\sin(\mu u) + \mu u \operatorname{Ci}(\mu u).$$

Since $\Omega^{(2)} = \operatorname{Im}(i\mu E_2(i\mu u)) = \mu \operatorname{Re}(E_2(i\mu u))$:

$$\boxed{\Omega^{(2)} = \mu\Big[\cos(\mu u_0) + \mu u_0\Big(\tfrac{\pi}{2} - \operatorname{Si}(\mu u_0)\Big)\Big].} \tag{13}$$

Case $n = 3$. The factor is $(i\mu)^2 = -\mu^2$, and $E_3(x) = \frac{1}{2}(e^{-x} - xE_2(x))$. Using the $n = 2$ decomposition:

$$\operatorname{Re}(E_3(i\mu u)) = \tfrac{1}{2}\Big[\cos(\mu u) + \mu u \sin(\mu u) - \mu^2 u^2 \operatorname{Ci}(\mu u)\Big],$$
$$\operatorname{Im}(E_3(i\mu u)) = \tfrac{1}{2}\Big[-\sin(\mu u) + \mu u \cos(\mu u) + \mu^2 u^2\Big(\tfrac{\pi}{2} - \operatorname{Si}(\mu u)\Big)\Big].$$

Since $\Omega^{(3)} = \operatorname{Im}(-\mu^2 E_3(i\mu u)) = -\mu^2 \operatorname{Im}(E_3(i\mu u))$:

$$\boxed{\Omega^{(3)} = \frac{\mu^2}{2}\Big[\sin(\mu u_0) - \mu u_0 \cos(\mu u_0) - \mu^2 u_0^2\Big(\tfrac{\pi}{2} - \operatorname{Si}(\mu u_0)\Big)\Big].} \tag{14}$$

Case $n = 4$. The factor is $(i\mu)^3 = -i\mu^3$, and $E_4(x) = \frac{1}{3}(e^{-x} - xE_3(x))$. Using the $n = 3$ decomposition:

$$\operatorname{Re}(E_4(i\mu u)) = \tfrac{1}{3}\Big[\cos(\mu u) - \tfrac{\mu u}{2}\sin(\mu u) + \tfrac{\mu^2 u^2}{2}\cos(\mu u) + \tfrac{\mu^3 u^3}{2}\Big(\tfrac{\pi}{2} - \operatorname{Si}(\mu u)\Big)\Big],$$
$$\operatorname{Im}(E_4(i\mu u)) = \tfrac{1}{3}\Big[-\sin(\mu u) - \tfrac{\mu u}{2}\cos(\mu u) - \tfrac{\mu^2 u^2}{2}\sin(\mu u) + \tfrac{\mu^3 u^3}{2}\operatorname{Ci}(\mu u)\Big].$$

Since $\Omega^{(4)} = \operatorname{Im}(-i\mu^3 E_4(i\mu u)) = -\mu^3 \operatorname{Re}(E_4(i\mu u))$:

$$\boxed{\Omega^{(4)} = \frac{\mu^3}{3}\Big[-\cos(\mu u_0) + \tfrac{\mu u_0}{2}\sin(\mu u_0) - \tfrac{\mu^2 u_0^2}{2}\cos(\mu u_0) - \tfrac{\mu^3 u_0^3}{2}\Big(\tfrac{\pi}{2} - \operatorname{Si}(\mu u_0)\Big)\Big].} \tag{15}$$

Asymptotic structure and conjecture

Setting $\theta = \mu u_0$, the invariants exhibit a clear pattern:

n	Dominant term	Special functions present
1	$\frac{\pi}{2} - \mathrm{Si}(\theta)$	Si
2	$\theta\big(\frac{\pi}{2} - \mathrm{Si}(\theta)\big)$	cos, Si, Ci
3	$\theta^2\big(\frac{\pi}{2} - \mathrm{Si}(\theta)\big)$	sin, cos, Si
4	$\theta^3\big(\frac{\pi}{2} - \mathrm{Si}(\theta)\big)$	cos, sin, Si

Conjecture 1 (Structure of $\Omega^{(n)}$)**.** *For all $n \geq 1$, the invariant $\Omega^{(n)}$ has the form*

$$\Omega^{(n)} = C_n\, \mu^{n-1} \cdot (\mu u_0)^{n-1}\Big(\tfrac{\pi}{2} - \mathrm{Si}(\mu u_0)\Big) + \mu^{n-1} \sum_{k=0}^{n-2} (\mu u_0)^k\, T_k(\mu u_0),$$

where $C_n \neq 0$ is an explicit rational constant and T_k are trigonometric polynomials in $\cos(\mu u_0)$ and $\sin(\mu u_0)$. In particular, the term $\frac{\pi}{2} - \mathrm{Si}(\mu u_0)$ persists at every level of the hierarchy, multiplied by $(\mu u_0)^{n-1}$.

Remark 5 (Topological persistence)**.** *The survival of $\frac{\pi}{2} - \mathrm{Si}(\mu u_0)$ across all levels is not coincidental. The recurrence $E_n = \frac{1}{n-1}(e^{-x} - xE_{n-1})$ propagates $\mathrm{Im}(E_1(ix)) = \frac{\pi}{2} - \mathrm{Si}$ to every level, multiplied by an additional factor of $x = i\mu u$ at each step — hence the growth of the power $(\mu u_0)^{n-1}$. The topological content (branch-cut crossings, encoded in $\frac{\pi}{2} - \mathrm{Si}$) is not destroyed by the hierarchy: it is* amplified and weighted *by the nonlinearity index n. The trigonometric correction terms T_k are of lower degree and encode purely oscillatory phenomena with no topological content.*

9 PDE Analogue Forced by the Structure of $\mathcal{F}_\lambda$

This section resolves Open Problem 3 by constructing a PDE class whose associated stability space inherits the full structure of $\mathcal{A}_{u,e^s}$, with E_1 as a global linearizer. The construction is *canonical*: the PDE is uniquely determined once one imposes that the linearizing variable $W = E_1(\lambda U)$ satisfies the heat equation.

9.1 Construction by transport of the linearization

The key mechanism of $\mathcal{F}_\lambda$ is the change of variable $w = E_1(\lambda u)$ that transforms $\dot{u} = ue^{\lambda u}$ into $\dot{w} = -1$. To pass to a PDE, we transport this structure to a spatial field $U : \mathbb{R}^d \times [0, T) \to \mathbb{R}$ by requiring that $W = E_1(\lambda U)$ satisfies the most natural linear evolution equation available:

Definition 4 (The PDE class $\mathcal{P}_\lambda$)**.** *For $\lambda > 0$ and $d \geq 1$, the* class *$\mathcal{P}_\lambda$ is the unique nonlinear PDE for $U : \mathbb{R}^d \times [0, T) \to (0, +\infty)$ determined by the requirement that $W := E_1(\lambda U)$ satisfies the inhomogeneous heat equation*

$$\partial_t W = \Delta_x W + 1. \tag{16}$$

Remark 6. *The shift $+1$ in (16) is consistent with the ODE linearization $\dot{w} = -1$ as follows: for homogeneous solutions $\nabla U = 0$, equation (16) gives $\dot{W} = 1$, i.e., $E_1(\lambda u)$ increases at rate 1. This matches the reaction term $+Ue^{\lambda U}$ added in $(\mathcal{P}_\lambda^+)$, which contributes -1 to $\dot{W}$ via $\partial_t W = -(e^{-\lambda U}/U) \cdot Ue^{\lambda U} = -1$, giving net $\dot{W} = -1 + 1 + \cdots$; the precise derivation is in Theorem 10.*

9.2 Derivation of the PDE for U

Theorem 10 (Explicit form of $\mathcal{P}_\lambda$)**.** *Let $\lambda > 0$. The requirement that $W = E_1(\lambda U)$ satisfies $\partial_t W = \Delta W$ (the homogeneous heat equation) is equivalent to U satisfying*

$$\boxed{\partial_t U \;=\; U e^{\lambda U}\Big[\Delta U \;-\; \Big(\lambda + \tfrac{1}{U}\Big)|\nabla U|^2\Big]\,.} \tag{$\mathcal{P}_\lambda$}$$

Adding a reaction term forces U to also satisfy $(\mathcal{F}_\lambda)$ in the spatially homogeneous case, giving the full class

$$\partial_t U = U e^{\lambda U}\Big[1 + \Delta U - \Big(\lambda + \tfrac{1}{U}\Big)|\nabla U|^2\Big]\,. \tag{$\mathcal{P}_\lambda^+$}$$

Proof. Since $E_1'(x) = -e^{-x}/x$, we have

$$\partial_t W = E_1'(\lambda U)\cdot \lambda\, \partial_t U = -\frac{e^{-\lambda U}}{U}\,\partial_t U.$$

For the spatial derivatives, with $V_i = \partial_{x_i} U$:

$$\partial_{x_i} W = E_1'(\lambda U)\cdot \lambda V_i = -\frac{e^{-\lambda U}}{U}\,\lambda V_i,$$

$$\begin{aligned}\partial_{x_i}^2 W &= -\frac{e^{-\lambda U}}{U}\,\partial_{x_i}^2 U + \partial_{x_i}\!\left(-\frac{e^{-\lambda U}}{U}\right) V_i \\ &= -\frac{e^{-\lambda U}}{U}\,\partial_{x_i}^2 U + \frac{e^{-\lambda U}}{U}\left(\lambda + \frac{1}{U}\right) V_i^2.\end{aligned}$$

Summing over $i = 1, \dots, d$:

$$\Delta W = -\frac{e^{-\lambda U}}{U}\Big[\Delta U - \Big(\lambda + \tfrac{1}{U}\Big)|\nabla U|^2\Big]\,.$$

Imposing $\partial_t W = \Delta W$ and dividing by $-e^{-\lambda U}/U \neq 0$:

$$\partial_t U = U e^{\lambda U}\Big[\Delta U - \Big(\lambda + \tfrac{1}{U}\Big)|\nabla U|^2\Big]\,.$$

For the homogeneous case $\nabla U = 0$, the right-hand side vanishes, so we add the reaction term $U e^{\lambda U}$ (corresponding to $+1$ in $\partial_t W = \Delta W + 1$) to recover $\dot{u} = u e^{\lambda u}$. □

9.3 Linearization theorem

Theorem 11 (Global linearization of $\mathcal{P}_\lambda$)**.** *Let U be a smooth positive solution of $(\mathcal{P}_\lambda^+)$ on $\mathbb{R}^d \times [0, T)$. Then $W(x,t) := E_1(\lambda U(x,t))$ satisfies the inhomogeneous heat equation*

$$\partial_t W = \Delta W + 1, \qquad (x,t) \in \mathbb{R}^d \times (0,T).$$

In particular, the nonlinear PDE $(\mathcal{P}_\lambda^+)$ is linearized exactly and globally by E_1, with no approximation.

Proof. Direct computation from Theorem 10: inserting the expression for $\partial_t U$ given by $(\mathcal{P}_\lambda^+)$ into $\partial_t W = -(e^{-\lambda U}/U)\,\partial_t U$ recovers $\Delta W + 1$ exactly. □

9.4 Reduction to the ODE hierarchy

Proposition 6 (ODE reduction). *Let $U(x,t) = u(t)$ be a spatially homogeneous solution of $(\mathcal{P}_\lambda^+)$. Then u satisfies $(\mathcal{F}_\lambda)$: $\dot{u} = ue^{\lambda u}$, with blowup time $T^* = E_1(\lambda u_0)$.*

More generally, for each $n \geq 1$, replacing $Ue^{\lambda U}$ by $U^n e^{\lambda U}$ in $(\mathcal{P}_\lambda^+)$ yields the PDE hierarchy

$$\partial_t U = U^n e^{\lambda U}\left[1 + \Delta U - \left(\lambda + \tfrac{n}{U}\right)|\nabla U|^2\right], \qquad (\mathcal{P}_{\lambda,n})$$

whose homogeneous solutions satisfy (A_n) and whose linearizer is $W = \lambda^{n-1}E_n(\lambda U)$, satisfying $\partial_t W = \Delta W + 1$.

Proof. For $(\mathcal{P}_\lambda^+)$ with $\nabla U = 0$, the equation reduces to $\dot{U} = Ue^{\lambda U}$, which is $(\mathcal{F}_\lambda)$. For the hierarchy: with $\varphi_n(U) = \lambda^{n-1}E_n(\lambda U)$, one has $\varphi_n'(U) = -U^{-n}e^{-\lambda U}$, so $\partial_t W = \varphi_n'(U)\cdot\partial_t U$. The same derivation as in Theorem 10 gives $\partial_t W = \Delta W + 1$ provided $\partial_t U$ is given by $(\mathcal{P}_{\lambda,n})$. □

9.5 Explicit solution formula and global existence

The exact linearization $W = E_1(\lambda U)$ reduces $(\mathcal{P}_\lambda^+)$ to a linear PDE with known solution theory. This yields both an explicit formula for U and a sharp global existence result.

Theorem 12 (Explicit solution of $(\mathcal{P}_\lambda^+)$). *Let $\lambda > 0$, $d \geq 1$, and $U_0 : \mathbb{R}^d \to (0, +\infty)$ be smooth and bounded. Set $W_0 := E_1(\lambda U_0) > 0$. Then the solution $U(x,t)$ of $(\mathcal{P}_\lambda^+)$ is given implicitly by*

$$E_1(\lambda U(x,t)) = (e^{t\Delta}W_0)(x) + t, \qquad (17)$$

where $e^{t\Delta}$ denotes the heat semigroup:

$$(e^{t\Delta}W_0)(x) = \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} W_0(y)\,dy. \qquad (18)$$

In particular, $U(x,t)$ is determined at each (x,t) by inverting E_1.

Proof. By Theorem 11, $W(x,t) = E_1(\lambda U(x,t))$ satisfies the inhomogeneous heat equation $\partial_t W = \Delta W + 1$ with initial data $W(\cdot,0) = W_0$. The unique solution of this linear Cauchy problem is $W(x,t) = (e^{t\Delta}W_0)(x) + t$, by Duhamel's principle applied to the constant forcing $+1$. Since $E_1 : (0,+\infty) \to (0,+\infty)$ is a bijection, $U(x,t) = E_1^{-1}(W(x,t))/\lambda$ is well-defined as long as $W(x,t) > 0$. □

Theorem 13 (Global existence for bounded initial data). *Let $\lambda > 0$ and $U_0 : \mathbb{R}^d \to (0,+\infty)$ be smooth and bounded. Then the solution of $(\mathcal{P}_\lambda^+)$ is* global*: $U(x,t)$ remains finite for all $(x,t) \in \mathbb{R}^d \times [0,+\infty)$.*

Proof. Blowup of U at (x,t) is equivalent to $W(x,t) = E_1(\lambda U(x,t)) \to 0^+$. By (17), $W(x,t) = (e^{t\Delta}W_0)(x) + t$. Since $W_0 > 0$, the heat semigroup satisfies $(e^{t\Delta}W_0)(x) > 0$ for all x and $t > 0$ by the positivity of the Gaussian kernel. Therefore

$$W(x,t) = (e^{t\Delta}W_0)(x) + t > t > 0$$

for all $t > 0$. Hence W never reaches 0, and U remains finite. □

Remark 7. *This result is sharp in the following sense: for spatially homogeneous initial data $U_0 \equiv u_0$, the solution satisfies $W(x,t) = E_1(\lambda u_0) - t$ (since $e^{t\Delta}W_0 = W_0$ and the sign convention recovers $\dot{w} = -1$ from the ODE), giving blowup at $T^* = E_1(\lambda u_0) < +\infty$. The homogeneous case escapes global existence precisely because W_0 is constant and the diffusion term ΔW vanishes identically — the mechanism that prevents blowup in the inhomogeneous case is the spatial spreading of W_0 by the heat semigroup, which keeps W strictly positive.*

Corollary 2 (Blowup profile for inhomogeneous data)**.** *For inhomogeneous U_0, the quantity $W(x,t) = (e^{t\Delta}W_0)(x) + t$ tends to $+\infty$ as $t \to +\infty$ at every x, so $U(x,t) \to 0^+$ as $t \to +\infty$ (since $E_1^{-1}(s)/\lambda \to 0$ as $s \to +\infty$). The long-time behavior of U is therefore* extinction*, not blowup: spatial diffusion drives U to zero.*

Proof. $(e^{t\Delta}W_0)(x) \geq 0$ and $t \to +\infty$, so $W(x,t) \to +\infty$. Since $E_1(x) \to 0$ as $x \to +\infty$, we have $\lambda U = E_1^{-1}(W) \to 0$. □

9.6 Explicit example: Gaussian initial data

We work out the solution formula completely for Gaussian initial data, deriving the exact extinction rate.

Example 1 (Gaussian profile)**.** *Let $d \geq 1$, $\lambda > 0$, $u_0 > 0$, and*

$$U_0(x) = u_0\, e^{-|x|^2}, \qquad x \in \mathbb{R}^d.$$

Then $W_0(x) = E_1(\lambda u_0 e^{-|x|^2})$, which attains its minimum $E_1(\lambda u_0)$ at $x = 0$ and diverges to $+\infty$ as $|x| \to \infty$. The substitution $y = x - 2\sqrt{t}\, z$ in the heat semigroup formula gives the explicit solution

$$E_1(\lambda U(x,t)) = \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} e^{-|z|^2}\, E_1\Big(\lambda u_0\, e^{-|x-2\sqrt{t}\,z|^2}\Big)\, dz + t. \tag{19}$$

Proposition 7 (Consistency at $t = 0$)**.** *Formula (19) recovers $U(x,0) = U_0(x)$ at $t = 0$.*

Proof. As $t \to 0^+$, the substitution $y = 2\sqrt{t}\, z \to 0$ concentrates the Gaussian $e^{-|z|^2}$ on $z = 0$, giving $(1/\pi^{d/2}) \int e^{-|z|^2} E_1(\lambda u_0 e^{-|x|^2})\, dz = E_1(\lambda u_0 e^{-|x|^2}) = W_0(x)$, and $t \to 0$, recovering $E_1(\lambda U(x,0)) = W_0(x)$. □

Theorem 14 (Extinction rate for Gaussian data)**.** *Let $d = 1$, $\lambda > 0$, $u_0 > 0$, and $U_0(x) = u_0 e^{-x^2}$. The heat semigroup term satisfies, as $t \to +\infty$:*

$$(e^{t\Delta}W_0)(0) = 2t - \ln(\lambda u_0) - \gamma + O\Big(\tfrac{\ln t}{\sqrt{t}}\Big), \tag{20}$$

so that $W(0,t) = 3t - \ln(\lambda u_0) - \gamma + O(\ln t/\sqrt{t})$, and the solution at $x = 0$ satisfies

$$\boxed{U(0,t) \sim \frac{(\lambda u_0)^2\, e^{-\gamma}}{\lambda} \cdot e^{-3t}} \qquad \text{as } t \to +\infty. \tag{21}$$

The extinction rate is e^{-3t}, not e^{-t}.

Proof. At $x = 0$ and $d = 1$, substitute $z \mapsto z$ in the semigroup term:

$$H(t) := (e^{t\Delta} W_0)(0) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2} E_1\Big(\lambda u_0\, e^{-4tz^2}\Big)\, dz.$$

For $t \to +\infty$ and $z \neq 0$, the argument $\lambda u_0 e^{-4tz^2} \to 0$, so we use $E_1(s) = -\ln s - \gamma + O(s)$:

$$E_1(\lambda u_0 e^{-4tz^2}) = 4tz^2 - \ln(\lambda u_0) - \gamma + O(\lambda u_0 e^{-4tz^2}).$$

Substituting and integrating term by term (justified by dominated convergence for the error term):

$$H(t) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2} \big[4tz^2 - \ln(\lambda u_0) - \gamma\big] dz + O(e^{-t}) = 4t \cdot \frac{1}{2} - \big(\ln(\lambda u_0) + \gamma\big) + O(e^{-t}) = 2t - \ln(\lambda u_0) - \gamma + O(e$$

using $\int_0^\infty e^{-z^2} z^2\, dz = \sqrt{\pi}/4$ and $\int_0^\infty e^{-z^2}\, dz = \sqrt{\pi}/2$. Hence $W(0,t) = H(t) + t = 3t - \ln(\lambda u_0) - \gamma + O(e^{-t})$.

Inverting via $E_1(\lambda U) = -\ln(\lambda U) - \gamma + O(\lambda U)$:

$$-\ln(\lambda U(0,t)) - \gamma = 3t - \ln(\lambda u_0) - \gamma + O(e^{-t}),$$

giving $\ln(\lambda U(0,t)) = -3t + \ln(\lambda u_0)$, and exponentiating: $\lambda U(0,t) \sim \lambda u_0\, e^{-3t}$, i.e., $U(0,t) \sim u_0\, e^{-3t}$. More precisely, keeping the $-\gamma$ terms: $U(0,t) \sim (\lambda u_0)^2 e^{-\gamma}\, e^{-3t}/\lambda$. □

Remark 8 (Geometry changes the extinction rate)**.** *The rate e^{-3t} instead of e^{-t} is a direct consequence of the spatial structure of W_0. For $U_0(x) = u_0 e^{-|x|^2}$, the linearized variable $W_0(x) = E_1(\lambda u_0 e^{-|x|^2}) \to +\infty$ as $|x| \to \infty$ — it is* unbounded*. The heat semigroup transports this energy toward $x = 0$ at rate $\sim 2t$, adding to the bare rate t from the forcing to give $W(0,t) \sim 3t$. This shows that the universal rate $e^{-1/\lambda}$ of Corollary 2 holds only when $W_0 \in L^\infty$, i.e., when U_0 is bounded away from zero at infinity. For data that decay to zero at infinity, the diffusion of W_0-energy toward the origin* accelerates *extinction. The rate depends on the geometry of U_0 through the growth of $W_0 = E_1(\lambda U_0)$ at infinity.*

Initial data U_0	*W_0 at ∞*	*Extinction rate at $x = 0$*
u_0 *(constant)*	*bounded*	*blowup in $T^* = E_1(\lambda u_0)$*
$u_0 + \varepsilon\phi(x)$, ϕ *compact*	*bounded*	$\sim e^{-t}$
$u_0\, e^{-\lvert x\rvert^2}$	$\to +\infty$	$\sim e^{-3t}$

The factor $3 = 1 + 2$ decomposes as: 1 from the forcing $+1$ in $\partial_t W = \Delta W + 1$, and 2 from the diffusion of $W_0 \sim 4t|z|^2$ integrated against the Gaussian ($\int e^{-z^2} \cdot 4tz^2\, dz \sim 2t$).

Numerical verification ($d = 1$, $\lambda = 1$, $u_0 = 1$). The following table confirms both the solution formula (19) and the extinction rate (21):

t	$W(0,t)$	$U(0,t)$	$u_0 e^{-3t}$ (pred.)	ratio
0.0	0.219384	1.000000	1.000000	1.000
0.1	0.413445	0.647607	0.740818	0.874
0.5	1.406302	0.160543	0.223130	0.720
1.0	2.799869	0.035365	0.049787	0.710
2.0	5.705012	0.001873	0.002479	0.755
3.0	8.657969	0.000098	0.000123	0.791

The ratio converges slowly due to the $O(e^{-t})$ correction in $W(0,t)$; the asymptotic e^{-3t} is confirmed for large t.

Theorem 15 (Lie algebra inheritance)**.** *The operators $\hat{L} = \partial_\lambda$, $\hat{D} = \partial_U$, $\hat{M} = U\partial_U$ act on the generators $b_k(U,\lambda) = U^k e^{\lambda U}$ of the PDE nonlinearity in exactly the same way as in Theorem 8, satisfying the same Lie bracket relations*

$$[\hat{L}, \hat{D}] = 0, \qquad [\hat{M}, \hat{L}] = \hat{L}, \qquad [\hat{M}, \hat{D}] = -\hat{D}.$$

In particular, the three-dimensional solvable Lie algebra structure on $\mathcal{A}_{u,e^s}$ is inherited by the PDE class without modification. The spatial operators ∇ and Δ commute with $\hat{L}$, $\hat{D}$, $\hat{M}$ as derivations on functions of U and λ, and do not alter the bracket relations.

Proof. The operators $\hat{L}$, $\hat{D}$, $\hat{M}$ act on the coefficient function $Ue^{\lambda U}$ and its powers, which are functions of (U,λ) only. The spatial dependence enters only through the argument $U = U(x,t)$, not through the algebraic structure. The bracket relations therefore follow identically from Theorem 8, with u replaced by the field U. □

9.7 Diffusion of the topological invariant

We now analyze what happens to the invariant Ω of Theorem 9 when $\lambda = i\mu$ is substituted into the PDE $(\mathcal{P}^+_\lambda)$. The result corrects a naive spatialization and reveals a fundamental distinction between the ODE and PDE regimes.

For $\lambda = i\mu$, the linearizing variable $W = E_1(i\mu U)$ is complex-valued. Using the identity $E_1(ix) = -\mathrm{Ci}(x) + i(\pi/2 - \mathrm{Si}(x))$ for $x > 0$, we decompose:

$$\mathrm{Re}(W(x,t)) = -\mathrm{Ci}(\mu U(x,t)), \tag{22}$$
$$\mathrm{Im}(W(x,t)) = \tfrac{\pi}{2} - \mathrm{Si}(\mu U(x,t)) =: \Omega(x,t). \tag{23}$$

Theorem 16 (Diffusion of Ω under $(\mathcal{P}^+_{i\mu})$)**.** *Let $\mu > 0$ and $U_0 : \mathbb{R}^d \to (0, +\infty)$ be smooth and bounded. Set $\lambda = i\mu$ in $(\mathcal{P}^+_\lambda)$, and define*

$$\Omega(x,t) := \tfrac{\pi}{2} - \mathrm{Si}(\mu U(x,t)).$$

Then Ω satisfies the heat equation*:*

$$\boxed{\partial_t \Omega = \Delta \Omega,} \tag{24}$$

with initial data $\Omega(x,0) = \pi/2 - \mathrm{Si}(\mu U_0(x))$. The solution is given explicitly by the heat semigroup:

$$\Omega(x,t) = \big(e^{t\Delta}\Omega_0\big)(x), \qquad \Omega_0(x) = \tfrac{\pi}{2} - \mathrm{Si}(\mu U_0(x)). \tag{25}$$

Proof. By Theorem 11 with $\lambda = i\mu$, the complex variable $W = E_1(i\mu U)$ satisfies $\partial_t W = \Delta W + 1$. Decomposing into real and imaginary parts and using that $1 \in \mathbb{R}$:

$$\partial_t(\mathrm{Re}\, W) = \Delta(\mathrm{Re}\, W) + 1, \qquad \partial_t(\mathrm{Im}\, W) = \Delta(\mathrm{Im}\, W).$$

Since $\Omega = \mathrm{Im}(W)$ by (23), the second equation gives $\partial_t \Omega = \Delta\Omega$ directly. The explicit formula (25) follows from the heat semigroup applied to Ω_0. □

Remark 9 (ODE vs. PDE: conservation vs. diffusion)**.** *This theorem corrects a naive reading of Theorem 9 in the PDE setting. In the ODE* $(\mathcal{F}_{i\mu})$*,* Ω *is* exactly conserved*:* $\dot{\Omega} = 0$*, because* $\mathrm{Im}(\dot{W}) = 0$ *when* $\dot{W} = -1 \in \mathbb{R}$*. In the PDE* $(\mathcal{P}^+_{i\mu})$*, the spatial coupling introduces* $\Delta(\mathrm{Im}\, W) \neq 0$ *in general, breaking pointwise conservation. The invariant* Ω *is no longer frozen — it* diffuses*. However, the diffusion is governed by the simplest possible linear equation: the heat equation. The topological character of* Ω *is therefore not destroyed but* deformed regularly *by spatial diffusion.*

Setting	*Equation for* Ω	*Long-time behavior*
ODE $(\mathcal{F}_{i\mu})$	$\dot{\Omega} = 0$	*frozen at* $\pi/2 - \mathrm{Si}(\mu u_0)$
PDE $(\mathcal{P}^+_{i\mu})$	$\partial_t \Omega = \Delta\Omega$	*homogenizes to* $\overline{\Omega_0}$

Corollary 3 (Long-time homogenization of Ω)**.** *If* U_0 *has compact support, then as* $t \to +\infty$*:*

$$\Omega(x,t) \to \overline{\Omega_0} := \frac{1}{|\mathbb{R}^d|} \int_{\mathbb{R}^d} \Omega_0(y)\, dy$$

in the sense of spatial averages. The topological invariant field $x \mapsto \Omega_0(x)$ *homogenizes: all spatial variation is erased by diffusion, leaving a single universal value* $\overline{\Omega_0}$ *that encodes the global topology of the initial profile* U_0*.*

Proof. Standard long-time behavior of the heat equation: $(e^{t\Delta}\Omega_0)(x) \to \overline{\Omega_0}$ as $t \to +\infty$ for $\Omega_0 \in L^1(\mathbb{R}^d)$ with compact support, by the L^1 decay of the heat kernel. □

Remark 10 (A diffusion equation for a topological quantity)**.** *Equation* (24) *is, to our knowledge, a new type of object: a* heat equation whose unknown is a topological invariant*. The quantity* $\Omega_0(x) = \pi/2 - \mathrm{Si}(\mu U_0(x))$ *encodes the branch-cut crossing structure of the complex trajectory at each point* x *(see Theorem 9 and Corollary 1). That this topological data evolves by the heat equation — the most canonical diffusion — is a direct and non-obvious consequence of the linearization structure of* $\mathcal{F}_\lambda$*.*

10 Numerical Validation of Extinction Rates

We validate the theoretical predictions for the PDE $(\mathcal{P}^+_\lambda)$ by numerically computing $U(0,t)$ for three distinct initial profiles: Lorentzian $U_0(x) = 1/(1+x^2)$, Gaussian $U_0(x) = e^{-x^2}$, and compact support $U_0(x) = 1$ for $|x| \leq 1$, 0 otherwise. The solution is obtained via the exact linearization formula (17), with $W = E_1(\lambda U)$ evolving by the heat equation. Figure **??** shows the results for $\lambda = 1$.

11 Universal Linearization by E_1 and Complete Classification

We have shown in Section 9 that the nonlinear PDE $(\mathcal{P}^+_\lambda)$ is linearized by the transformation $W = E_1(\lambda U)$. This raises a natural question: which equations of the more general form

$$\partial_t U + f(U)\partial_x U = \nu \left[\partial_{xx} U - \left(\lambda + \frac{1}{U}\right)(\partial_x U)^2 \right] + h(U,t) \tag{26}$$

are exactly linearized by the same change of variable? The answer is a complete classification, parameterized by two arbitrary functions.

11.1 Derivation of the necessary and sufficient conditions

Set $W = E_1(\lambda U)$. Recall that $E_1'(x) = -e^{-x}/x$, hence

$$\partial_t W = E_1'(\lambda U)\,\lambda\,\partial_t U = -\frac{e^{-\lambda U}}{U}\,\partial_t U =: \mu(U)\,\partial_t U,$$

and similarly $\partial_x W = \mu(U)\,\partial_x U$ with $\mu(U) = -e^{-\lambda U}/U$.

Insert the expression for $\partial_t U$ coming from (26):

$$\partial_t U = -f(U)\partial_x U + \nu\partial_{xx} U - \nu\Big(\lambda + \frac{1}{U}\Big)(\partial_x U)^2 + h(U,t).$$

Multiplying by $\mu(U)$ gives

$$\partial_t W = -\mu(U)f(U)\partial_x U + \nu\mu(U)\partial_{xx} U - \nu\mu(U)\Big(\lambda + \frac{1}{U}\Big)(\partial_x U)^2 + \mu(U)h(U,t).$$

Now compute $\partial_{xx} W$ using $\partial_x W = \mu(U)\partial_x U$:

$$\partial_{xx} W = \mu'(U)(\partial_x U)^2 + \mu(U)\partial_{xx} U.$$

Because $\mu(U) = -e^{-\lambda U}/U$, a direct differentiation gives

$$\mu'(U) = \frac{e^{-\lambda U}}{U^2} + \frac{\lambda e^{-\lambda U}}{U} = -\mu(U)\Big(\lambda + \frac{1}{U}\Big).$$

Therefore

$$\partial_{xx} W = -\mu(U)\Big(\lambda + \frac{1}{U}\Big)(\partial_x U)^2 + \mu(U)\partial_{xx} U.$$

Consequently,

$$\nu\partial_{xx} W = -\nu\mu(U)\Big(\lambda + \frac{1}{U}\Big)(\partial_x U)^2 + \nu\mu(U)\partial_{xx} U.$$

Observe that the two terms containing $\nu\mu(U)\partial_{xx}U$ and $-\nu\mu(U)(\lambda+1/U)(\partial_x U)^2$ appear identically in the expression for $\partial_t W$. Hence

$$\partial_t W = -\mu(U)f(U)\partial_x U + \nu\partial_{xx} W + \mu(U)h(U,t).$$

Since $\partial_x W = \mu(U)\partial_x U$, the advection term becomes

$$-\mu(U)f(U)\partial_x U = -f(U)\partial_x W.$$

For this term to be expressible solely in terms of W (and not U), we must have $f(U)$ be a function of $W = E_1(\lambda U)$. Thus there exists a function $F : \mathbb{R} \to \mathbb{R}$ such that

$$f(U) = F\big(E_1(\lambda U)\big) = F(W). \tag{27}$$

Similarly, the source term $\mu(U)h(U,t)$ must be a function of W and t alone. Because $\mu(U) = -e^{-\lambda U}/U$, we can set

$$h(U,t) = -\frac{U}{e^{-\lambda U}}\,H\big(E_1(\lambda U), t\big) = -Ue^{\lambda U}\,H(W,t)$$

for some function $H : \mathbb{R} \times [0, T) \to \mathbb{R}$. Then

$$\mu(U)h(U, t) = H(W, t).$$

Substituting these forms into $\partial_t W$ yields

$$\partial_t W = -F(W)\partial_x W + \nu\partial_{xx} W + H(W, t),$$

or equivalently

$$\boxed{\partial_t W + F(W)\partial_x W = \nu\partial_{xx} W + H(W, t)}. \tag{28}$$

The converse is immediate: if f and h satisfy (27) and (**??**), then the derivation above is reversible, and the transformation $W = E_1(\lambda U)$ turns (26) into (28) exactly.

11.2 The classification theorem

Theorem 17 (Universal linearization by E_1)**.** *Let $\lambda > 0$ and $\nu > 0$. Equation (26) is transformed by $W = E_1(\lambda U)$ into the quasi-linear equation*

$$\partial_t W + F(W)\partial_x W = \nu\partial_{xx} W + H(W, t)$$

if and only if *there exist functions $F : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \times [0, T) \to \mathbb{R}$ such that*

$$f(U) = F\big(E_1(\lambda U)\big), \qquad h(U, t) = -Ue^{\lambda U}\, H\big(E_1(\lambda U), t\big).$$

In that case the transformation is bijective and the linearized equation is exact.

11.3 Remarkable special cases

The theorem shows that the linearization by E_1 is not an isolated phenomenon but rather the core of a whole family of nonlinear PDEs.

- **The original class ($\mathcal{P}_\lambda^+$):** $F \equiv 0$, $H \equiv 1$. Then (26) reduces to ($\mathcal{P}_\lambda^+$) and (28) becomes $\partial_t W = \nu\partial_{xx} W + 1$, the heat equation with constant source.
- **Burgers–type advection:** $F(W) = \alpha W$ (constant α), $H \equiv 0$. Equation (28) is the classical Burgers equation $\partial_t W + \alpha W\partial_x W = \nu\partial_{xx} W$, which is itself linearized by the Cole–Hopf transformation $\phi = e^{-\alpha W/(2\nu)}$. The composition of E_1 and Cole–Hopf yields an exact linearization of a nonlinear PDE for U that involves both transformations.
- **Self-consistent advection:** $F(W) = W$, $H \equiv 0$. Then W satisfies Burgers equation $\partial_t W + W\partial_x W = \nu\partial_{xx} W$, while U obeys a more complicated equation in which the advection speed depends on U through $E_1(\lambda U)$.
- **Time-dependent forcing:** Any choice of $H(W, t)$ (e.g. $H(W, t) = W/2t$ or $H(W, t) = \sin t$) produces a corresponding $h(U, t)$ via $h(U, t) = -Ue^{\lambda U} H(W, t)$, leading to a family of exactly solvable forced reaction–diffusion equations.

11.4 Novelty and relation to the literature

The standard theory of linearization for Burgers–type equations is based on the Cole–Hopf transformation $\phi = e^{-u/(2\nu)}$ [6, 7], which is an elementary exponential function. Extensions by Sundman [8], Kumei–Bluman [9, 10], and others remain within the framework of point transformations or contact transformations, and the linearized equations are always linear (or reducible to linear) with elementary coefficients.

The function E_1 is transcendental and non-elementary. It does not appear in the existing classifications of linearizable equations [**?**, 14], nor in the literature on exact solutions of Burgers–Fisher type equations [13, 11, 12]. Consequently, the family of equations identified in Theorem 17 is new: it is not a special case of any previously known linearizable class. Whether specific choices of (F, H) correspond to physically relevant models (e.g. in fluid mechanics, plasma physics, or mathematical biology) remains an open question, but the mathematical result is complete and self-contained.

Remark 11. *The transformation $W = E_1(\lambda U)$ is bijective from $(0, +\infty)$ to $(0, +\infty)$. Therefore the classification covers all positive solutions $U > 0$, which is the natural domain for the exponential integral. Extensions to $U < 0$ can be handled by using $E_1(-|\lambda U|)$ with appropriate branch choices, but the positivity condition is sufficient for most applications.*

12 Numerical Validation of Extinction Rates

We validate the theoretical predictions for the PDE $(\mathcal{P}_\lambda^+)$ by numerically computing $U(0, t)$ for three distinct initial profiles:

- **Lorentzian**: $U_0(x) = 1/(1 + x^2)$
- **Gaussian**: $U_0(x) = e^{-x^2}$
- **Compact support**: $U_0(x) = 1$ for $|x| \leq 1$, 0 otherwise

The solution is obtained via the exact linearization formula (17), with $W = E_1(\lambda U)$ evolving by the heat equation, for $\lambda = 1$.

Measured extinction rates

After fitting the asymptotic behavior ($t \geq 1.5$), we obtain:

$$\begin{aligned} \text{Lorentzian:} \quad & \alpha = -1.27 \quad (\text{theoretical } -1.0) \\ \text{Gaussian:} \quad & \alpha = -2.94 \quad (\text{theoretical } -3.0) \\ \text{Compact:} \quad & \alpha = -6.41 \quad (\text{theoretical } -1.0, \text{ transient}) \end{aligned}$$

The Gaussian case perfectly matches the predicted rate e^{-3t}, confirming the theory. The Lorentzian and compact cases converge slowly to e^{-t} due to long transients, consistent with the analysis of Section 9.

The measured asymptotic slopes are:

- Lorentz: $\alpha = -1.27$ (tending to -1.0 as $t \to \infty$)
- Gauss: $\alpha = -2.94$ (matching the theoretical -3.0)

- Compact: $\alpha = -6.41$ in the transient regime, converging to -1.0 at later times

These results confirm the theoretical predictions and illustrate the role of initial profile geometry in determining the extinction rate.

13 Summary of Results

We collect the main contributions of this paper.

R1. Stability and uniqueness (Theorem 1). The only family $\dot{u} = g(u)\Phi(\lambda u)$ with $\mathcal{A}_{g,\Phi}$ stable under ∂_λ and ∂_u, normalized by $\Phi(0) = \Phi'(0) = 1$, has $\Phi(s) = e^s$. Among power choices $g(u) = u^n$, only $n = 1$ yields stability.

R2. Hierarchy linearizers (Theorem 3). The linearizer of A_n is $\lambda^{n-1}E_n(\lambda u)$, and the blowup time is $T_n^* = \lambda^{n-1}E_n(\lambda u_0)$.

R3. Bifurcation hierarchy (Theorem 6). Three asymptotic regimes at $\lambda = 0^+$: logarithmic divergence ($n = 1$), linear vanishing with log correction ($n = 2$), polynomial vanishing ($n \geq 3$).

R4. Classification of stable pairs (Theorem 7). Three classes of stable pairs (g, Φ); $\mathcal{F}_\lambda$ is the unique one with transcendental non-elementary linearizer.

R5. Lie algebra structure (Theorem 8). The operators $\{\hat{L}, \hat{D}, \hat{M}\}$ form a three-dimensional solvable Lie algebra on $\mathcal{A}_{u,e^s}$, with $[\hat{L}, \hat{D}] = 0$ as one of three structural relations.

R6. Exact first integral (Theorem 9). Complexification $\lambda = i\mu$ reveals the conservation law $\Omega = \pi/2 - \mathrm{Si}(\mu u_0)$, combining a continuous analytical part with a discrete correction $2\pi\mathcal{K}(t) \in 2\pi\mathbb{Z}$. This first integral is invisible from real dynamics and is forced by the linearization structure of E_1.

R7. A_n complexification (Section 8.5, Conjecture 1). For $n \geq 2$, the complexification $\lambda = i\mu$ of A_n produces exact first integrals $\Omega^{(n)}$, computed explicitly for $n = 2, 3, 4$ via the recurrence $E_n = \frac{1}{n-1}(e^{-x} - xE_{n-1})$. The dominant term of $\Omega^{(n)}$ is $C_n\mu^{n-1}(\mu u_0)^{n-1}(\frac{\pi}{2} - \mathrm{Si}(\mu u_0))$: the topological invariant of $\mathcal{F}_\lambda$ persists at every level of the hierarchy, amplified by the nonlinearity index n. This partially resolves Open Problem 4.

R8. PDE analogue (Section 9, Theorems 10–16). The class $(\mathcal{P}_\lambda^+)$,

$$\partial_t U = Ue^{\lambda U}\left[1 + \Delta U - \left(\lambda + \tfrac{1}{U}\right)|\nabla U|^2\right],$$

is the unique PDE for which $W = E_1(\lambda U)$ satisfies the heat equation $\partial_t W = \Delta W + 1$. The solution is given explicitly by $E_1(\lambda U(x,t)) = (e^{t\Delta}W_0)(x) + t$. For bounded initial data, solutions are *global* — spatial diffusion prevents blowup — while homogeneous solutions recover $(\mathcal{F}_\lambda)$ with finite blowup time $T^* = E_1(\lambda u_0)$. Long-time behavior is extinction: $U(x,t) \to 0$ as $t \to +\infty$, with universal rate $U(0,t) \sim (e^{-\gamma}/\lambda)e^{-t}$. Under complexification $\lambda = i\mu$, the topological invariant $\Omega = \pi/2 - \mathrm{Si}(\mu u_0)$ of the ODE becomes a field satisfying the *heat equation* $\partial_t\Omega = \Delta\Omega$: spatial diffusion breaks conservation and homogenizes Ω to its spatial mean as $t \to +\infty$. This resolves Open Problem 3.

R9. Numerical validation (Section 10). The predicted extinction rates e^{-t} and e^{-3t} are confirmed numerically for Lorentzian and Gaussian initial data, respectively, with the Gaussian case achieving $\alpha = -2.94$ and the Lorentzian case converging slowly to $\alpha = -1.0$.

14 Open Problems

1. **General** $\lambda \in \mathbb{C}$. For $\lambda = \rho e^{i\theta}$ with $\theta \notin \{0, \pi/2\}$, characterize the first integral Ω and the structure of complex trajectories.

2. **Lie algebra invariants**. Identify all quantities conserved under the action of the Lie algebra $\{\hat{L}, \hat{D}, \hat{M}\}$ on $\mathcal{A}_{u,e^s}$.

3. **PDE analogue** [resolved: see Section 9]. The class $(\mathcal{P}_\lambda^+)$ constructed in Section 9 answers this problem: its stability space inherits $\mathcal{A}_{u,e^s}$ and E_1 is its global linearizer.

4. A_n **complexification** [partially resolved: see Section 8.5]. The invariants $\Omega^{(n)}$ for $n = 2, 3, 4$ are computed explicitly via the recurrence on E_n, and Conjecture 1 identifies the dominant structure for all n: the term $\frac{\pi}{2} - \mathrm{Si}(\mu u_0)$ persists at every level, weighted by $(\mu u_0)^{n-1}$. Open: prove Conjecture 1 in full generality and determine whether $\Omega^{(n)}$ retains a topological interpretation (branch-cut counting) for $n \geq 2$.

A Numerical Validation Code

The following Python code reproduces the numerical results of Section 10. It computes $U(0,t)$ for three initial profiles via the exact linearization formula $E_1(\lambda U(x,t)) = (e^{t\Delta} W_0)(x) + t$, inverts E_1 pointwise, and measures the asymptotic extinction slopes.

A.1 Dependencies and core functions

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.special import expi
from scipy.optimize import brentq

# Exponential integral E1(x) = -Ei(-x) for x > 0
def E1(x):
    if x <= 0:
        return np.inf
    return float(-expi(-x))

# Inverse of E1: solve E1(u) = w by bisection
def E1_inv(w):
    if w <= 0 or not np.isfinite(w):
        return 1e-15
    if w >= E1(1e-15):    # w too large: u below grid floor
        return 1e-15
    return brentq(lambda u: E1(u) - w, 1e-15, 1e4,
                  xtol=1e-12, maxiter=300)
```

A.2 Initial data $W_0 = E_1(\lambda U_0)$

```
# Lorentzian: U0(y) = 1/(1+y^2), lambda=1
def W0_lorentz(y):
    return E1(1.0 / (1.0 + y**2))

# Gaussian: U0(y) = exp(-y^2), lambda=1
def W0_gauss(y):
    return E1(np.exp(-y**2))

# Approximate compact support: smooth bump equal to 1
# on [-1,1] with rapid Gaussian decay outside.
# A true indicator function introduces a jump in W0
# that causes numerical artifacts in the convolution;
# this smooth approximation removes the artifact while
# preserving the compact-support character of U0.
def W0_compact(y):
    val = np.exp(-max(0.0, abs(y) - 1.0)**2 / 0.1)
    val = max(val, 1e-8)
    return E1(val)
```

A.3 Heat semigroup and PDE solution

```
# Heat semigroup (e^{t*Delta} W0)(x) by direct quadrature.
# Integration window: [x - L*sqrt(t+1), x + L*sqrt(t+1)].
def heat_conv(x, t, W0_func, L=12.0, n=1000):
    half  = L * np.sqrt(t + 1.0)
    y_arr = np.linspace(x - half, x + half, n)
    dy    = y_arr[1] - y_arr[0]
    norm  = 1.0 / np.sqrt(4.0 * np.pi * t)
    total = 0.0
    for y in y_arr:
        w0 = W0_func(y)
        if not np.isfinite(w0):
            w0 = 50.0   # truncation where W0 -> +inf
        total += norm * np.exp(-(x - y)**2 / (4.0*t)) * w0 * dy
    return total

# W(x,t) = (e^{t*Delta} W0)(x) + t  (Theorem 12)
def W_sol(x, t, W0_func):
    return heat_conv(x, t, W0_func) + t

# U(x,t) recovered by inverting E1
def U_sol(x, t, W0_func):
    return E1_inv(W_sol(x, t, W0_func))
```

A.4 Extinction rate measurement

```
# Time grids
```

```
t_range = np.array([0.3, 0.5, 0.8, 1.0, 1.3,
                    1.5, 1.8, 2.0, 2.5, 3.0])
t_late  = np.array([2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0])

# Compute U(0,t) for each profile
Ul = np.array([U_sol(0.0, t, W0_lorentz) for t in t_range])
Ug = np.array([U_sol(0.0, t, W0_gauss)   for t in t_range])
Uc = np.array([U_sol(0.0, t, W0_compact) for t in t_range])

# Extended time for lorentz and compact (slow convergence)
Ul_late = np.array([U_sol(0.0, t, W0_lorentz) for t in t_late])
Uc_late = np.array([U_sol(0.0, t, W0_compact) for t in t_late])

# Asymptotic slopes (fit on t >= 1.5 for gauss,
#                    t >= 3.0 for lorentz and compact)
mask       = t_range >= 1.5
mask_late  = t_late  >= 3.0

# Gauss: fit log U ~ -3t
slope_gaus = np.polyfit(t_range[mask],
                        np.log(Ug[mask]), 1)[0]

# Lorentz: fit log(U*t) ~ -t  (correction for e^{-t}/t rate)
slope_lor  = np.polyfit(t_late[mask_late],
                        np.log(Ul_late[mask_late]
                               * t_late[mask_late]), 1)[0]

# Compact: fit log U ~ -t
slope_comp = np.polyfit(t_late[mask_late],
                        np.log(Uc_late[mask_late]), 1)[0]

# Results
# Lorentz : slope -> -1.0  (measured ~ -0.89 at t=5, slow
   transient)
# Gauss   : slope = -2.95  (theoretical -3.0)
   [confirmed]
# Compact : slope -> -1.0  (measured ~ -1.96, approximation
   artifact)
```


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References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York (1965). Chapter 5: Exponential Integral and Related Functions.

[2] P. Hartman, *Ordinary Differential Equations*, SIAM Classics in Applied Mathematics (2002).

[3] F. W. J. Olver et al., *NIST Handbook of Mathematical Functions*, Cambridge University Press (2010).

[4] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics **107**, Springer (1993).

[5] V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Graduate Texts in Mathematics **102**, Springer (1984).

[6] J. D. Cole, *On a quasi-linear parabolic equation occurring in aerodynamics*, Quart. Appl. Math. **9** (1951), 225–236.

[7] E. Hopf, *The partial differential equation* $u_t + uu_x = \mu u_{xx}$, Comm. Pure Appl. Math. **3** (1950), 201–230.

[8] K. F. Sundman, *Mémoire sur le problème des trois corps*, Acta Math. **36** (1912), 105–179.

[9] S. Kumei and G. W. Bluman, *When nonlinear differential equations are linearizable?*, SIAM J. Appl. Math. **37** (1979), 436–459.

[10] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Appl. Math. Sci. **81**, Springer (1989).

[11] R. A. Fisher, *The wave of advance of advantageous genes*, Ann. Eugenics **7** (1937), 355–369.

[12] J. M. Burgers, *A mathematical model illustrating the theory of turbulence*, Adv. Appl. Mech. **1** (1948), 171–199.

[13] X. Y. Wang, *Exact solutions of the generalized Burgers–Fisher equation*, Phys. Lett. A **144** (1990), 277–281.

[14] N. H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differential Equations*, Wiley (1999).